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On the Theory of Newforms of Half-Integral Weight

M. MANICKAM,* B. RAMAKRISHNAN,* AND T. C. VASUDEVAN

Department of Mathematics, R. K. M. Vivekananda College, Mylapore, Madras 600 004, India

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In this paper we will study the theory of newforms in $S_{k+1/2}(\Gamma_0(4M), \chi_1)$, for M an odd squarefree natural number, $\chi_1 = (4\varepsilon/\cdot)\,\chi$, where χ is a quadratic character modulo M with $\chi(-1)=\varepsilon$, parallel to the Atkin–Lehner theory of newforms in $S_{2k}(2M)$ and prove that these two spaces are isomorphic under certain linear combinations of Shimura liftings, commuting with the action of Hecke operators.

1. INTRODUCTION

In his famous paper [6], G. Shimura introduced the space of modular forms of half-integral weight whose level is a natural number divisible by 4 and constructed certain liftings of the space of cusp forms of half-integral weight to the space of cusp forms of integral weight; these liftings commute with the action of Hecke operators. Later S. Niwa [4] proved that the levels of such forms of integral weight are equal to half the levels of the forms of half-integral weight. Further, S. Niwa [5] proved that the trace of the Hecke operator $T(n^2)$ on $S_{k+1/2}(\Gamma_0(4M))$ is equal to the trace of the Hecke operator $T(n)$ on $S_{2k}(2M)$, where M is an odd squarefree natural number and $(n, 2M)=1$.

W. Kohnen [3] introduced a subspace, called the "+ space," of the space of cusp forms of half-integral weight and studied the theory of newforms parallel to the Atkin–Lehner theory of newforms of integral weight. He also proved the existence of a linear combination of Shimura liftings which maps the newform space of the +space isomorphically onto the space of newforms of integral weight, whose level is an odd squarefree natural number. For this purpose he proved, as a main tool, that the trace of the Hecke operator $T(n^2)$ on $S^+_{k+1/2}(\Gamma_0(4M))$ is equal to the trace of the

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Hecke operator $T(n)$ on $S_{2k}(M)$, where M is an odd squarefree natural number and $(n, 2M)=1$.

In this paper we consider the space $S_{k+1/2}(\Gamma_0(4M), \chi_1)$ for M an odd squarefree natural number, $\chi_1=(4\varepsilon/\cdot)\,\chi$, $\chi(-1)=\varepsilon$, where χ is a quadratic character modulo M. We study the theory of newforms in that space and prove the existence of a linear combination of Shimura liftings which is an isomorphism of the spaces $S_{k+1/2}(\Gamma_0(4M), \chi_1)$ and $S_{2k}(2M)$, commuting with the action of Hecke operators, using only the equality of the traces proved by S. Niwa [5]. We note that Theorem 2 of W. Kohnen [3] (for $k\geqslant 2$) follows as a consequence of our results.

2. Notations

Let k, N be natural numbers, $k\geqslant 2$ and N squarefree. Set

$$M=\begin{cases} N/2 & \text{if } N \text{ is even} \\ N & \text{if } N \text{ is odd.}\end{cases}$$

Let p denote a prime and d denote a positive divisor of N. By $a\,|\,b$, we understand that a is a positive divisor of b. Let χ be a quadratic character modulo M with conductor t; let $\chi_1=(4\varepsilon/\cdot)\,\chi$, where $\chi(-1)=\varepsilon$.

We denote by $S_{k+1/2}(\Gamma_0(4M), \chi_1)$ the space of cusp forms of weight $k+1/2$ and character χ_1 for the group $\Gamma_0(4M)$ in Shimura's sense and by $S^+_{k+1/2}(\Gamma_0(4M), \chi_1)$, the Kohnen $+$space which is the subspace of $S_{k+1/2}(\Gamma_0(4M), \chi_1)$ consisting of cusp forms whose nth Fourier coefficient vanishes whenever $\varepsilon(-1)^k n\equiv 2, 3 \pmod 4$.

Put

$$S_{k+1/2}(N, \chi)=\begin{cases} S_{k+1/2}(\Gamma_0(4M), \chi_1) & \text{if } N \text{ is even} \\ S^+_{k+1/2}(\Gamma_0(4M), \chi_1) & \text{if } N \text{ is odd.}\end{cases}$$

We write

$$S_{k+1/2}(N, 1)=S_{k+1/2}(N).$$

$S_{2k}(N)$ denotes the space of cusp forms of weight $2k$ for $\Gamma_0(N)$ with trivial character. For details the reader is referred to [2, 3, 6].

For a natural number m, the operator $U(m)$ is defined on formal power series in x by

$$\sum_{n\geqslant 1} a(n)\,x^n\,|\,U(m)=\sum_{n\geqslant 1} a(mn)\,x^n,$$

where

$$x = e^{2\pi i z}, \qquad \mathrm{Im} \quad z > 0.$$

The analogous Atkin–Lehner W-operators on $S_{k+1/2}(N, \chi)$ are defined as follows: For $p \mid M$,

$$W(p) = \left(\begin{pmatrix} pa & b \\ 4Mc & p \end{pmatrix}, p^{-1/4} (4Mcz + p)^{1/2} \right),$$

where $b \equiv 1 \pmod{p}$ and a, b, c are integers satisfying $p^2 a - 4Mbc = p$;

$$W(4) = \left(\begin{pmatrix} 4a & b \\ 4Mc & 4 \end{pmatrix}, \varepsilon^{-1/2} e^{\pi i/4} (-1)^{[(k+1)/2]}\, 2^{1/2} (Mcz + 1)^{1/2} \right)$$

where $b \equiv 1 \pmod{4}$ and a, b, c are integers satisfying $16a - 4Mbc = 4$. Note that the operator $W(p)$, $p \mid M$ is (up to a constant factor equal to ± 1) the W-operator considered by W. Kohnen [3]. For $p \mid N$, denote by W_p the Atkin–Lehner W-operator on $S_{2k}(N)$ given by

$$W_p = \begin{pmatrix} pa & b \\ Nc & pd \end{pmatrix},$$

where a, b, c, d are integers satisfying $p^2 ad - Nbc = p$.

If $p \nmid N$ is odd, denote by $T(p^2)$ the Shimura Hecke operator on $S_{k+1/2}(N, \chi)$ and define

$$T(4) = \tfrac{3}{2} U(4)\, \mathrm{pr},$$

where pr is the orthogonal projection onto $S_{k+1/2}(M, \chi)$ given by

$$\mathrm{pr} = \tfrac{2}{3}(Q - \beta) \qquad \text{(see [3, p. 42])}.$$

For a form f, we denote by $a_f(n)$ the nth Fourier coefficient of f and for a number theoretic function $a(n)$, we put $a(n) = 0$ if n is not an integer.

Then, for $f \in S_{k+1/2}(N, \chi)$ and $p \nmid N$, we have

$$f \mid T(p^2) = \sum_{n \geq 1} \left(a_f(np^2) + \chi(p) \left(\frac{\varepsilon(-1)^k n}{p} \right) p^{k-1} a_f(n) + p^{2k-1} a_f(n/p^2) \right) x^n.$$

Note that the operator $T(p^2)$, $p \nmid N$ preserves the space $S_{k+1/2}(N, \chi)$ (cf. [3, 6]).

$T(p)$, $p \nmid N$ denotes the Hecke operator on $S_{2k}(N)$. For $f \in S_{2k}(N)$ and $p \nmid N$, we have

$$f \mid T(p) = \sum_{n \geq 1} \{ a_f(np) + p^{2k-1} a_f(n/p) \}\, x^n.$$

If f and g are cusp forms of weight $k+1/2$ for a subgroup Γ of finite index in $\Gamma_0(4)$, we define the Petersson scalar product of f and g by

$$\langle f, g\rangle = \frac{1}{[\Gamma_0(4):\Gamma]}\int_{\Gamma\backslash H} f(z)\,\overline{g(z)}\,v^{k-3/2}\,du\,dv,$$

where H is the upper half-plane and $z=u+iv\in H$.

For a fundamental discriminant D (i.e., D equals 1 or is the discriminant of a quadratic field) with $\varepsilon(-1)^k D>0$, define a map $\mathscr{S}_{D,k,M,\chi}$ by

$$\sum_{n\geqslant 1} b(n)\,x^n \mid \mathscr{S}_{D,k,M,\chi} = \sum_{n\geqslant 1}\left\{\sum_{r\mid n}\chi(r)\left(\frac{D}{r}\right) r^{k-1} b(|D|\,n^2/r^2)\right\} x^n.$$

Then $\mathscr{S}_{D,k,M,\chi}$ maps the space $S_{k+1/2}(N,\chi)$ to the space $S_{2k}(N)$. When χ is trivial, we denote the map by $\mathscr{S}_{D,k,M}$(of [3]). Note that when $D\equiv 0 \pmod{M}$,

$$\mathscr{S}_{D,k,1}=\mathscr{S}_{D,k,d}, \qquad d \text{ odd on } S_{k+1/2}(2d). \tag{2.1}$$

3. Theory of Newforms on $S_{k+1/2}(N)$

Lemma 1. *For $p\mid M$, $W(p)$ is an isomorphism of the spaces $S_{k+1/2}(N,\chi)$ and $S_{k+1/2}(N,(\cdot/p)\,\chi)$.*

Proof. If N is odd, this is Proposition 2 of [3]. If N is even, the proof is clear, since $W(p)$ maps the space $S_{k+1/2}(N,\chi)$ to the space $S_{k+1/2}(N,(\cdot/p)\,\chi)$.

Lemma 2. *$U(t)$ is an isomorphism of the spaces $S_{k+1/2}(N)$ and $S_{k+1/2}(N,\chi)$.*

Proof. Since the proof of Proposition 3 of [3] holds also for N even, the proof is clear.

Let

$$w_p=\begin{cases} p^{-k/2+1/4}\,U(p)\,W(p) & \text{if } p \text{ odd}\\ W(4) & \text{if } p \text{ even.}\end{cases}$$

Lemma 3. *For $p\mid N$, w_p is a hermitean involution on $S_{k+1/2}(N)$.*

Proof. Note that for $p\mid M$, the proof of Proposition 4 of [3] also holds for N even. For $p=2$, it is easy to check that $W(4)^2=\text{Identity}$.

Definition. A non-zero form $f\in S_{k+1/2}(N)$ is called a Hecke eigenform if it is an eigenform with respect to all $T(p^2)$ $(p\nmid N)$ and $U(p^2)$ $(p\mid N)$ for odd p and with respect to $T(4)$ or $U(4)$ accordingly as f is in the plus space or not.

LEMMA 4. *$U(4)\,W(4) = (-1)^{[(k+1)/2]}\,\varepsilon 2^{k-3/2} Q$ on $S_{k+1/2}(N, \chi)$, where Q is the operator defined in* [3].

Proof. For u, v mod 4, let

$$\alpha_u = \left(\begin{pmatrix} 1 & u \\ 0 & 4 \end{pmatrix}, 2^{1/2}\right) W(4) \qquad \text{and} \qquad \beta_v = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \varepsilon^{1/2} e^{\pi i/4}\right) \begin{pmatrix} 1 & 0 \\ 4Mv & 1 \end{pmatrix}^*.$$

Then an easy computation shows that

$$\alpha_u = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \varepsilon^{-1}(-1)^{[(k+1)/2]}\right) C^* \beta_v, \qquad \text{where} \quad C \in \Gamma_0(4M). \tag{3.1}$$

Since

$$U(4)\,W(4) = 2^{k-3/2} \sum_{u \bmod 4} \alpha_u$$

and

$$Q = \sum_{v \bmod 4} \beta_v \qquad (\text{cf. [3, p. 36]}),$$

(3.1) implies that

$$U(4)\,W(4) = (-1)^{[(k+1)/2]}\,\varepsilon 2^{k-3/2} Q \qquad \text{on} \quad S_{k+1/2}(N, \chi).$$

Define the space of oldforms of $S_{k+1/2}(N)$ by

$$S^{\text{old}}_{k+1/2}(N) = \sum_{d<N} \left(S_{k+1/2}(d) + S_{k+1/2}(d) \mid U(N^2/d^2)\right)$$

and the space of newforms, denoted by $S^{\text{new}}_{k+1/2}(N)$, to be the orthogonal complement of $S^{\text{old}}_{k+1/2}(N)$ in $S_{k+1/2}(N)$ under Petersson scalar product.

THEOREM 1. *For $p \mid N$, the operators $U(p^2)$ and w_p preserve $S^{\text{new}}_{k+1/2}(N)$ and $U(p^2) = -p^{k-1} w_p$ on $S^{\text{new}}_{k+1/2}(N)$*

Proof. If p is odd, the proof is in line with the proof of Theorem 1 of [3].

Let $p = 2$. (In this case N is even). By Lemma 4, the projection map is given by

$$3\text{pr} = 2^{1-k} U(4)\,W(4) + 1 \qquad (\text{cf. [3]}). \tag{3.2}$$

Let $f \in S^{\text{new}}_{k+1/2}(N)$ Then, by definition,

$$f \mid \text{pr} = 0,$$

i.e.,

$$f \mid (2^{1-k} U(4)\, W(4) + 1) = 0.$$

Applying $W(4)$ and using the fact that $W(4)^2 = 1$, we have

$$U(4) = -2^{k-1} W(4) \qquad \text{on} \quad S^{\text{new}}_{k+1/2}(N).$$

It remains to prove that $W(4)$ preserves $S^{\text{new}}_{k+1/2}(N)$. Since $W(4)$ is hermitean, it suffices to prove that $W(4)$ preserves $S^{\text{old}}_{k+1/2}(N)$. For this purpose, by (3.2), it suffices to prove that $U(4)$ preserves $S^{\text{old}}_{k+1/2}(N)$. Since

$$\sum_{\substack{d \mid N \\ d \text{ even}}} (S_{k+1/2}(d) + S_{k+1/2}(d) \mid U(N^2/d^2)) \mid U(4) \subseteq S^{\text{old}}_{k+1/2}(N)$$

and

$$S_{k+1/2}(M) \mid U(4) \subseteq S^{\text{old}}_{k+1/2}(N),$$

we have to show that

$$S_{k+1/2}(M) \mid U(4)^2 \subseteq S^{\text{old}}_{k+1/2}(N).$$

By (3.2), we have

$$T(4) = W(4)\, U(4)\, W(4) + 2^{-1} U(4) \qquad \text{on} \quad S_{k+1/2}(M).$$

Let $g \in S_{k+1/2}(M)$. Then

$$g \mid T(4) = g \mid (W(4)\, U(4)\, W(4) + 2^{-1} U(4)) \in S_{k+1/2}(M).$$

Applying $W(4)$ and using Lemmas 3 and 4, we have

$$g \mid W(4)\, U(4) \in S_{k+1/2}(M) + S_{k+1/2}(M) \mid U(4),$$

i.e.,

$$g \mid U(4)^2 \in S_{k+1/2}(M) + S_{k+1/2}(M) \mid U(4) \subseteq S^{\text{old}}_{k+1/2}(N),$$

which completes the proof.

THEOREM 2. *For N even,*

$$\operatorname{tr}(T(n^2), S_{k+1/2}(N)) = \operatorname{tr}(T(n), S_{2k}(N))$$

for $(n, N) = 1$.

Proof. See Remark 4 of [5]. Also one can refer to [7, Theorem p. 541] for a detailed proof.

THEOREM 3. *There exists a basis of Hecke eigenforms for* $S_{k+1/2}^{\text{new}}(N)$.

Proof. Since the Hecke operator $T(p^2)$, $p \nmid N$ is hermitean and preserves $S_{k+1/2}^{\text{new}}(N)$, the theorem follows from Lemma 3 and Theorem 1.

LEMMA 5. *Let* $f \in S_{k+1/2}(d)$ *be a Hecke eigenform. Then there exists a fundamental discriminant* $D \equiv 0 \pmod{8M}$ *with* $(-1)^k D > 0$ *such that* $a_f(|D|) \neq 0$.

Proof. Assume the contrary.
Consider the function

$$g = f \mid U(4M) \in S_{k+1/2}\left(N, \left(\frac{\cdot}{M}\right)\right).$$

By assumption, we have

$$a_g(n) = 0 \qquad \text{whenever } n \equiv 2 \pmod 4.$$

Therefore,

$$f \mid U(4M) \in S_{k+1/2}\left(M, \left(\frac{\cdot}{M}\right)\right) \qquad \text{(cf. [3, Lemma, p. 69])}.$$

Using Lemma 2, we have

$$f \mid U(4) \in S_{k+1/2}(M)$$

i.e.,

$$f \mid U(4) \in \begin{cases} S_{k+1/2}(d/2), & d \ \text{even} \\ S_{k+1/2}(d), & d \ \text{odd}. \end{cases} \tag{3.3}$$

Let d be even.

Case (i). $f \in S_{k+1/2}(d/2)$. Then

$$f \mid T(4) = \lambda_2 f;$$

i.e,

$$\tfrac{3}{2} f \mid U(4)\,\text{pr} = \lambda_2 f \qquad \text{or} \qquad \tfrac{3}{2} f \mid U(4) = \lambda_2 f \qquad (\text{by } (3.3)).$$

Applying $W(4)$ and using Lemma 4, we have

$$3 \cdot 2^{k-1} f = \lambda_2 f \mid W(4),$$

which implies that

$$|\lambda_2| = 3 \cdot 2^{k-1},$$

a contradiction to Deligne's theorem.

Case ii. $f \in S_{k+1/2}(d/2)$. Then

$$f \mid U(4) = \lambda_2 f \in S_{k+1/2}(d/2) \qquad \text{(by (3.3))}$$

a contradiction.

The case d odd is nothing but Case (i), which proves the lemma completely.

THEOREM 4. *The space $S_{k+1/2}(N)$ can be decomposed as*

$$S_{k+1/2}(N) = \bigoplus_{rd \mid N} S^{\text{new}}_{k+1/2}(d) \mid U(r^2). \tag{3.4}$$

Proof. For $N=2$, the decomposition is clear using Case (i) of Lemma 5. We now prove the decomposition for the case $N=2q$, q an odd prime. We prove the general case by induction.

By Theorem 3, let $\{f_1, ..., f_{r_1}\}$, $\{f_{r_1+1}, ..., f_{r_2}\}$, $\{f_{r_2+1}, ..., f_{r_3}\}$, and $\{f_{r_3+1}, ..., f_{r_4}\}$ be bases of Hecke eigenforms for $S_{k+1/2}(1)$, $S^{\text{new}}_{k+1/2}(2)$, $S^{\text{new}}_{k+1/2}(q)$, and $S^{\text{new}}_{k+1/2}(2q)$, respectively. By Lemma 5, there exist fundamental discriminants $D_i \equiv 0 \pmod{8q}$ with $(-1)^k D_i > 0$ such that $a_{f_i}(|D_i|) \neq 0$, $1 \leqslant i \leqslant r_4$.

Then the polynomial

$$P(X_1, ..., X_{r_4}) = \prod_{1 \leqslant i \leqslant r_4} \left(\sum_{j=1}^{r_4} a_{f_i}(|D_j|) X_j \right)$$

is non-zero. Choosing constants $\alpha_1, ..., \alpha_{r_4}$ such that $P(\alpha_1, ..., \alpha_{r_4}) \neq 0$, we define a map

$$\mathscr{S}_{k,2q} = \sum_{i=1}^{r_4} \alpha_i \mathscr{S}_{D_i,k,1}.$$

Then

$$F_i = f_i \mid \mathscr{S}_{k,2q} \neq 0, \qquad 1 \leqslant i \leqslant r_4.$$

Also, $\mathscr{S}_{k,2q}$ maps the spaces $S_{k+1/2}(1)$, $S_{k+1/2}(2)$, $S_{k+1/2}(q)$, and $S_{k+1/2}(2q)$ to the spaces $S_{2k}(1)$, $S_{2k}(2)$, $S_{2k}(q)$, and $S_{2k}(2q)$, respectively (cf. (2.1)).

Since

$$T(p^2)\,\mathcal{S}_{D_i,k,1} = \mathcal{S}_{D_i,k,1}\,T(p), \qquad p \text{ odd}$$
$$T(4)\,\mathcal{S}_{D_i,k,1} = \mathcal{S}_{D_i,k,1}\,T(2) \qquad \text{on} \quad S_{k+1/2}(q)$$
$$U(p^2)\,\mathcal{S}_{D_i,k,1} = \mathcal{S}_{D_i,k,1}\,U(p)\,p\,|\,2q, \qquad 1 \leqslant i \leqslant r_4$$
$$f_i\,|\,U(q^2) = \pm q^{k-1} f_i, \qquad r_2 < i \leqslant r_4$$

and

$$f_i\,|\,U(4) = \pm 2^{k-1} f_i,\; r_1 < i \leqslant r_2,$$

the Atkin–Lehner newform theory shows that $\mathcal{S}_{k,2q}$ maps the spaces $S^{\text{new}}_{k+1/2}(2)$, $S^{\text{new}}_{k+1/2}(q)$, and $S^{\text{new}}_{k+1/2}(2q)$ to the spaces $S^{\text{new}}_{2k}(2)$, $S^{\text{new}}_{2k}(q)$, and $S^{\text{new}}_{2k}(2q)$, respectively.

By definition, we have

$$S_{k+1/2}(q) = S^{\text{new}}_{k+1/2}(q) \oplus (S_{k+1/2}(1) + S_{k+1/2}(1)\,|\,U(q^2))$$

and

$$S_{k+1/2}(2q) = S^{\text{new}}_{k+1/2}(2q) \oplus \Bigg(\sum_{\substack{rd\,|\,2q\\ d<2q}} S^{\text{new}}_{k+1/2}(d)\,|\,U(r^2) \Bigg).$$

For $1 \leqslant i \leqslant r_1$, let

$$f_i, f_i\,|\,U(q^2) \in S_{k+1/2}(1).$$

Then, applying the map $\mathcal{S}_{k,2q}$, we have $F_i, F_i\,|\,U(q) \in S_{2k}(1)$, a contradiction. Therefore

$$S_{k+1/2}(q) = S^{\text{new}}_{k+1/2}(q) \oplus S_{k+1/2}(1) \oplus S_{k+1/2}(1)\,|\,U(q^2).$$

Similarly, applying the map $\mathcal{S}_{k,2q}$ on $S^{\text{new}}_{k+1/2}(d)\,|\,U(r^2)$, one can prove that

$$\sum_{\substack{rd\,|\,2q\\ d<2q}} S^{\text{new}}_{k+1/2}(d)\,|\,U(r^2) = \bigoplus_{\substack{rd\,|\,2q\\ d<2q}} S^{\text{new}}_{k+1/2}(d)\,|\,U(r^2),$$

proving the required decomposition for $S_{k+1/2}(2q)$.

We now prove the analogue of Theorem 4 of [1].

THEOREM 5. *Let g_1 and g_2 be two newforms belonging to $S^{\text{new}}_{k+1/2}(N_1)$ and $S^{\text{new}}_{k+1/2}(N_2)$, respectively ($N_1$ and N_2 are squarefree natural numbers) such that g_1 and g_2 have the same eigenvalues with respect to infinitely many operators $T(p^2)$ for $(p, N_1N_2) = 1$. Then $N_1 = N_2$ and g_1 is a constant multiple of g_2. In particular, the "multiplicity 1" theorem holds on $S^{\text{new}}_{k+1/2}(N)$.*

Proof. Let $N_3 = \text{l.c.m. of } N_1, N_2$.

Since g_1 and g_2 are Hecke eigenforms, by Lemma 5, we can choose fundamental discriminants $D_i \equiv 0 \pmod{8N_3}$ or $D_i \equiv 0 \pmod{4N_3}$ (according as N_3 is odd or even) with $(-1)^k D_i > 0$, $i = 1, 2$. Then we define a map

$$\mathscr{S}_k^{N_3} = a_1 \mathscr{S}_{D_i, k, 1} + a_2 \mathscr{S}_{D_i, k, 1}, \qquad a_1, a_2 \in \mathbb{C},$$

satisfying

$$G_i = g_i \mid \mathscr{S}_k^{N_3} \neq 0, \qquad i = 1, 2$$

(by constructing a non-zero complex polynomial as in the above theorem).

By (2.1), it is clear that

$$G_i \in S_{2k}(N_i), \qquad i = 1, 2$$

Also

$$g_i \mid U(p^2) = \pm p^{k-1} g_i, \qquad p \mid N_i$$
$$T(p^2)\, \mathscr{S}_k^{N_3} = \mathscr{S}_k^{N_3} T(p), \qquad p \nmid N_i \text{ on } S_{k+1/2}(N_i),$$

which implies that

$$G_i \in S_{2k}^{\text{new}}(N_i), \qquad i = 1, 2, \text{ respectively} \tag{3.5}$$

(cf. [1, Theorem 5]).

Since G_1 and G_2 have the same eigenvalues of $T(p)$ for almost all p, it follows from Theorem 4 of [1] that

$$N_1 = N_2 \qquad (= N_3)$$

and

$$G_1 = G_2 \qquad (= G \text{ say}) \qquad \in S_{2k}^{\text{new}}(N_3)$$

(without loss of generality we can assume that G_1 and G_2 are normalized).

Case (i). N_3 even. By Theorem 2, there exists an isomorphism φ_1 from $S_{k+1/2}(N_3)$ onto $S_{2k}(N_3)$ such that

$$T(p^2)\, \varphi_1 = \varphi_1 T(p), \qquad p \nmid N_3.$$

Since $g_i \mid \varphi_1 \in S_{2k}(N_3)$ and $g_i \mid \varphi_1$, G have the same eigenvalues of $T(p)$, $p \nmid N_3$, it follows that

$$g_i \mid \varphi_1 \in \mathbb{C}G, \qquad i = 1, 2 \qquad \text{(cf. [1, Theorem 5])},$$

and hence g_1 is a constant multiple of g_2.

Case (ii). N_3 odd. Let φ_2 be an isomorphism (cf. Theorem 2) of the space $S_{k+1/2}(2N_3)$ onto the space $S_{2k}(2N_3)$ such that

$$T(p^2)\,\varphi_2 = \varphi_2 T(p), \qquad p \nmid 2N_3$$

Then $g_i|\varphi_2, g_i|U(4)|\varphi_2$, and G have the same eigenvalues of $T(p)$, $p \nmid 2N_3$, $i=1, 2$. Since $g_i|\varphi_2 \in S_{2k}(2N_3)$ by Atkin–Lehner newform theory,

$$g_i|\varphi_2, g_i|U(4)|\varphi_2 \in \mathbb{C}G \oplus \mathbb{C}G|U(2), \qquad i=1, 2.$$

If g_1 and g_2 are linearly independent, $g_1, g_2, g_1|U(4)$, and $g_2|U(4)$ are linearly independent and

$$(\mathbb{C}g_1 \oplus \mathbb{C}g_2 \oplus \mathbb{C}g_1|U(4) \oplus \mathbb{C}g_2|U(4))|\varphi_2 \subseteq (\mathbb{C}G \oplus \mathbb{C}G|U(2)),$$

a contradiction.

The proof is now complete.

THEOREM 6. *For any squarefree positive integer N the spaces $S^{\text{new}}_{k+1/2}(N)$ and $S^{\text{new}}_{2k}(N)$ are Hecke-equivariantly isomorphic via suitable linear combinations of the Shimura liftings.*

Proof. It suffices to prove the theorem for N even. Let $f_1, ..., f_s$ be the collection of all newforms belonging to $\bigcup_{r|N} S_{k+1/2}(r)$, where

$$s = \sum_{d|M} (R_d + S_d)$$

with

$$R_d = \dim S^{\text{new}}_{k+1/2}(2d) \qquad \text{and} \qquad S_d = \dim S^{\text{new}}_{k+1/2}(d).$$

For $d|M$, let

$$r_d = \dim S^{\text{new}}_{2k}(2d) \qquad \text{and} \qquad s_d = \dim S^{\text{new}}_{2k}(d).$$

By Lemma 5 we can choose fundamental discriminants $D_i \equiv 0 \pmod{8M}$ with $(-1)^k D_i > 0$ such that $a_{f_i}(|D_i|) \neq 0$. Then, by constructing a non-zero complex polynomial as in Theorem 4, we define a map

$$\mathscr{S}_{k,N} = \sum_{i=1}^{s} \alpha_i \mathscr{S}_{D_i,k,1}, \qquad \alpha_i \in \mathbb{C}$$

satisfying

$$f_i|\mathscr{S}_{k,N} \neq 0, \qquad 1 \leqslant i \leqslant s.$$

Then $\mathscr{S}_{K,N}$ maps the spaces $S_{k+1/2}(d)$ and $S_{k+1/2}(2d)$ to the spaces $S_{2k}(d)$ and $S_{2k}(2d)$, respectively (cf (2.1)). Since

$$\begin{aligned} T(p^2)\,\mathscr{S}_{k,N} &= \mathscr{S}_{k,N}\,T(p), \qquad p \text{ odd} \qquad (\text{cf. } (2.1)) \\ T(4)\,\mathscr{S}_{k,N} &= \mathscr{S}_{k,N}\,T(2), \qquad \text{on} \quad S_{k+1/2}(d) \\ U(p^2)\,\mathscr{S}_{k,N} &= \mathscr{S}_{k,N}\,U(p), \quad p \mid N \end{aligned}$$

and

$$f_i \mid U(p^2) = \pm p^{k-1} f_i, \qquad p \mid 2d \text{ or } p \mid d \text{ according as } f_i \in S^{\text{new}}_{k+1/2}(2d) \text{ or } S^{\text{new}}_{k+1/2}(d),$$

by the Atkin–Lehner newform theory, we have

$$S^{\text{new}}_{k+1/2}(2d) \mid \mathscr{S}_{k,N} \subseteq S^{\text{new}}_{2k}(2d)$$

and

$$S^{\text{new}}_{k+1/2}(d) \mid \mathscr{S}_{k,N} \subseteq S^{\text{new}}_{2k}(d).$$

By Theorem 5, the map $\mathscr{S}_{k,N}$ is one-to-one on $S^{\text{new}}_{k+1/2}(2d)$ and $S^{\text{new}}_{k+1/2}(d)$, which implies that

$$R_d \leqslant r_d \qquad \text{and} \qquad S_d \leqslant s_d. \tag{3.6}$$

We know that

$$S_{2k}(N) = \bigoplus_{rd \mid M} S^{\text{new}}_{2k}(2d) \mid U(r) \bigoplus_{\substack{rd \mid N \\ d \text{ odd}}} S^{\text{new}}_{2k}(d) \mid U(r). \tag{3.7}$$

Since the spaces $S_{k+1/2}(N)$ and $S_{2k}(N)$ are isomorphic (cf. Theorem 2), by (3.4) and (3.7), we have

$$\sum_{d \mid M} (\sigma_0(M/d)\, R_d + \sigma_0(N/d)\, S_d) = \sum_{d \mid M} (\sigma_0(M/d)\, r_d + \sigma_0(N/d)\, s_d) \tag{3.8}$$

($\sigma_0(n)$ denotes the number of positive divisors of n).

From (3.6) and (3.8), we can conclude that

$$R_d = r_d \qquad \text{and} \qquad S_d = s_d, \qquad d \mid M,$$

which completes the proof of the theorem.

Remark 1. Theorems 4, 5, 6 together with (3.7) imply that the map $\mathscr{S}_{k,N}$ is an isomorphism of the spaces $S_{k+1/2}(N)$ and $S_{2k}(N)$ such that

$$\begin{aligned} T(p^2)\,\mathscr{S}_{k,N} &= \mathscr{S}_{k,N}\,T(p), \qquad p \nmid N \\ U(p^2)\,\mathscr{S}_{k,N} &= \mathscr{S}_{k,N}\,U(p), \qquad p \mid N. \end{aligned}$$

Remark 2. Let $p \mid M$. Defining

$$S^{\pm,p}_{k+1/2}(N)=\left\{f\in S_{k+1/2}(N) \mid a_f(n)=0 \text{ for } \left(\frac{(-1)^k n}{p}\right)=\mp 1\right\}$$

one can prove that

$$S^{\pm,p}_{k+1/2}(N)=\{f\in S_{k+1/2}(N) \mid f \mid w_p=\pm f\} \qquad \text{(cf. [3, Proposition 4)]).}$$

Let

$$S^{\pm,p}_{2k}(N)=\{f\in S_{2k}(N) \mid f \mid W_p=\pm f\}.$$

By Theorem 1,

$$U(p^2)=-p^{k-1}w_p \qquad \text{on} \quad S^{\text{new}}_{k+1/2}(N)$$

and one can prove, on the lines of W. Kohnen [3, p. 66], that

$$T(p^2)=U(p^2)+p^{k-1}w_p \qquad \text{on} \quad S_{k+1/2}(N/p).$$

Note that

$$U(p)=-p^{k-1}W_p \qquad \text{on} \quad S^{\text{new}}_{2k}(N)$$

and

$$T(p)=U(p)+p^{k-1}W_p \qquad \text{on} \quad S_{2k}(N/p) \qquad \text{(cf. [1]).}$$

Also, by construction of the map $\mathscr{S}_{k,N}$,

$$T(p^2)\,\mathscr{S}_{k,N}=\mathscr{S}_{k,N}\,T(p) \qquad \text{on} \quad S_{k+1/2}(N/p),$$

and

$$U(p^2)\,\mathscr{S}_{k,N}=\mathscr{S}_{k,N}\,U(p), \qquad p \mid N.$$

Therefore, it follows that

$$w_p\mathscr{S}_{k,N}=\mathscr{S}_{k,N}W_p, \qquad p \mid M \text{ on } S_{k+1/2}(N). \tag{3.9}$$

Hence by Remark 1 and by (3.9) we see that the eigenspaces $S^{\pm,p}_{k+1/2}(N)$ and $S^{\pm,p}_{2k}(N)$ are isomorphic under $\mathscr{S}_{k,N}$ for $p \mid M$.

4. The Space $S_{k+1/2}(N,\chi)$

In this section we take χ non-trival.

An easy computation shows that for a fundamental discriminant $D\equiv 0 \pmod{8M}$ with $(-1)^k D>0$,

$$U(t)\,\mathscr{S}_{D_0,k,M,\chi}=\mathscr{S}_{D,k,d}, \qquad d \text{ odd}, \tag{4.1}$$

where $|D_0|t=|D|$ and D_0 is a fundamental discriminant with $\varepsilon(-1)^k D_0 > 0$.

We put

$$S_{k+1/2}(d,\chi)=S_{k+1/2}(d)|\,U(t) \qquad \text{(cf. Lemma 2)}.$$

Define

$$S^{\text{old}}_{k+1/2}(N,\chi)=\sum_{d<N}\left(S_{k+1/2}(d,\chi)+S_{k+1/2}(d,\chi)|\,U(N^2/d^2)\right)$$

and define $S^{\text{new}}_{k+1/2}(N,\chi)$ to be the orthogonal complement of $S^{\text{old}}_{k+1/2}(N,\chi)$ in $S_{k+1/2}(N,\chi)$ under Petersson scalar product.

LEMMA 6. $S^{\text{new}}_{k+1/2}(N,\chi)=S^{\text{new}}_{k+1/2}(N)|\,U(t)$.

Proof. The proof is the same as the proof of the Lemma of [3, p. 66]. The operator

$$w_{p,x}=\begin{cases} U(t)^{-1}\,w_p\,U(t) & p \text{ odd} \\ W(4) & p \text{ even}\end{cases}$$

is defined on $S^{\text{new}}_{k+1/2}(N,\chi)$ for $p\,|\,N$. We put

$$S^{\pm;p}_{k+1/2}(N,\chi)=S^{\pm;p}_{k+1/2}(N)|\,U(t).$$

Using the above facts alongwith Lemma 2, one can prove the results of Section 3 for χ non-trivial, which we summarise in the following.

THEOREM 7. (1) *The space $S_{k+1/2}(N,\chi)$ can be decomposed as*

$$S_{k+1/2}(N,\chi)=\bigoplus_{rd\,|\,N} S_{k+1/2}(d,\chi)|\,U(r^2).$$

(2) *The space $S^{\text{new}}_{k+1/2}(d,\chi)$ is an orthogonal direct sum of common eigenspaces of all $T(p^2)$, $p\nmid d$ and $U(p^2)$, $p\,|\,d$, each of dimension one. There is a linear combination of Shimura liftings, which is an isomorphism of the spaces $S^{\text{new}}_{k+1/2}(d,\chi)$, $S_{k+1/2}(N,\chi)$ onto the spaces $S^{\text{new}}_{2k}(d)$, $S_{2k}(N)$, respectively, given by*

$$\mathcal{S}_{k,N,\chi}=U(t)^{-1}\,\mathcal{S}_{k,N}$$

such that

$$T(p^2)\,\mathcal{S}_{k,N,\chi}=\mathcal{S}_{k,N,\chi}\,T(p), \qquad p\nmid N$$
$$U(p^2)\,\mathcal{S}_{k,N,\chi}=\mathcal{S}_{k,N,\chi}\,U(p), \qquad p\,|\,N.$$

(3) *$\mathcal{S}_{k,N,\chi}$ is an isomorphism of the eigenspaces $S^{\pm;p}_{k+1/2}(N,\chi)$ and $S^{\pm;p}_{2k}(N)$ for $p\,|\,M$.*

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